Vieta's Formulas and Newton Sums

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§1 Polynomials

In order to understand Vieta's Formulas, we need to make sure we understand some basic properties of polynomials.

Definition 1.1. A polynomial of degree n is represented by the expressions

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k, a_n \neq 0$$

One important theorem that pertains to polynomials is known as the Fundamental Theorem of Algebra.

Theorem 1.2 (Fundamental Theorem of Algebra)
All polynomials with complex coefficients with degree n have exactly n complex roots, counting multiplicity.

Note that this statement still applies for polynomials such as $(x-3)^2$ since it counts $x=3$ as two roots. The proof of this will not be explored here, but its impact is quite substantial. This will help form the basis of the formulas we will discuss.

§2 Vieta's Formulas on Quadratics

First, we explore Vieta's Formulas on polynomials of degree 2. We will generalize this later.

Theorem 2.1 (Vieta's Formulas on Quadratics)
Let the roots of the polynomial ax^2+bx+c be p and q. Then, $p+q = -\frac{b}{a}$ and $pq = \frac{c}{a}$.

Proof. Note that the polynomial can be written as $a(x-p)(x-q) = ax^2 - a(p+q)x + apq$. This implies that $-a(p+q) = b$ and $apq = c$, and the result follows. □

Vieta's Formulas are best learned through application, so let's try some problems.

Problem 2.2 — Given that α and β are the roots of $3x^2 - 50x + 63 = 0$, compute $\alpha+\beta$ and $\alpha\beta$.

Solution. By direct application of Vieta's Formulas, we see that $\alpha + \beta = -\frac{-50}{3} = \boxed{\frac{50}{3}}$ and $\alpha\beta = \frac{63}{3} = \boxed{21}$ □

This was fairly straightforward, but the true power of the Vieta's Formulas is that it allows us to find even complicated expressions without having to find any of the roots. See the below problem for an example.

Problem 2.3 — Given that α and β are the roots of $3x^2 - 50x + 63 = 0$, compute $\frac{1}{\alpha} + \frac{1}{\beta}$.

Solution. Let's combine the denominators first. We see that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$. We computed both the numerator and the denominator using Vieta's Formulas in the previous problem, so the answer is $\frac{\frac{50}{3}}{21} = \boxed{\frac{50}{63}}$. (A polynomial transformation would also work) □

§3 Vieta's Formulas Generalized

We can derive the formulas for greater degree polynomials in much the same way we did for second degree ones. Let's take a look at cubic polynomials for a moment.

Theorem 3.1 (Vieta's Formulas on Cubics)
Let the roots of the polynomial $ax^3 + bx^2 + cx + d$ be p, q, and r. Then, $p + q + r = -\frac{b}{a}$, $pq + qr + rp = \frac{c}{a}$, and $pqr = -\frac{d}{a}$.

The proof of this will be left as an exercise to the reader. As a hint, use the factored form of the cubic. Now with two examples of Vieta's Formulas, you may notice a pattern. As the number of roots being multiplied per term increases, the power of the x^k term corresponding to the coefficient used decreases. Thus, we derive the Generalized Vieta's Formulas as follows:

Theorem 3.2 (Vieta's Formulas Generalized)
Let the roots of the polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be $x_1, x_2, \ldots, x_n$. Then,

$$\sum_{\text{sym}} x_1x_2\cdots x_k = (-1)^k\frac{a_{n-k}}{a_n}$$

.

Note that $\sum_{\text{sym}} x_1x_2\cdots x_k$ represent the sum of all terms $x_{a_1}x_{a_2}\cdots x_{a_k}$ where $(a_1, a_2, \ldots, a_n)$ ranges over all combinations of $(1, 2, \ldots, k)$. A demonstration of this is shown below.

Problem 3.3 — Let the roots of the polynomial $2x^5 + 15x^4 + 10x^3 + 26x^2 + 50x - 1$ be a, b, c, d, e. Compute $abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde$.

Solution. Notice that the sum given is equal to $\sum_{\text{sym}} abc$ since the terms go through all possible combinations of (a, b, c, d, e). Thus, we'll just directly use the Generalized Vieta's Formulas to get that this sum is equal to $-\frac{a_2}{a_5} = -\frac{26}{2} = \boxed{-13}$. □

Problem 3.4 — Let the roots of the polynomial $2x^3 - 6x^2 + 10x + 16$ be a, b, c. Compute $a^3 + b^3 + c^3$.

Solution. We want to get this expression in terms of the Vieta's Formulas. Let's do some rearranging.

$$\begin{aligned} a^3 + b^3 + c^3 &= (a+b+c)^3 - 3(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2) - 6abc \\ &= (a+b+c)^3 - 3((a+b+c)(ab+bc+ca) - 3abc) - 6abc \\ &= (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc \end{aligned}$$

By Vieta's Formulas, we see that $a + b + c = 3$, $ab + bc + ca = 5$, $abc = -8$. Thus, we just plug these values in to get $3^3 - 3(3)(5) + 3(-8) = 27 - 45 - 24 = \boxed{-42}$ □

§4 Examples

Vieta's Formulas are one of the fundamental concepts in all algebra in competition math. It often appears as an intermediate step in a problem rather than a standalone concept, so being familiar with the formula is critical. Trying it out on some actual problems is a good way to get good at this.

Problem 4.1 — Prove Theorem 3.2

Solution. We can express the polynomial as $a_n(x-x_1)(x-x_2)\cdots(x-x_n)$. We can "choose" which of x or x_m we want to multiply with the rest of our terms to get our desired sum. That is, to get a combination of $x_1x_2\cdots x_k$, we must choose k x_m's and $(n-k)$ x's to multiply out. Thus, $(-1)^k a_n \sum_{\text{sym}} x_1x_2\cdots x_k$ will be the coefficient of the x^{n-k} term of the polynomial. Equating the coefficients completes the proof. □

Problem 4.2 (USMCA 2020 Challenger #12) — Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^2}{d^3+2}$$

.

Solution. We will attempt to simplify $\frac{a^2}{a^3+2}$. Note that since a is a root of $f(x)$, $a^4 + 2a + 4 = 0 \Rightarrow a^3 + 2 = -\frac{4}{a}$. Therefore,

$$\frac{a^2}{a^3+2} = -\frac{a^3}{4} = -\frac{1}{4}\left(-\frac{4}{a} - 2\right) = \frac{1}{a} + \frac{1}{2}$$

. By applying the argument with the rest of the roots, we obtain the expression $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 2 = \frac{abc+bcd+cda+dab}{abcd} + 2$. Notice that we can apply Vieta's Formulas on both the numerator and denominator. Thus,

$$\frac{abc+bcd+cda+dab}{abcd} + 2 = \frac{-2}{4} + 2 = \boxed{\frac{3}{2}}$$

□

Problem 4.3 (2019 AIME I #10) — For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. Compute the value of

$$\sum_{1 \le j < k \le 673} z_j z_k$$

Solution. Denote the value we need to compute as S. Note that S is equal to $\sum_{\text{sym}} z_1z_2$ over $(z_1, z_2, \ldots, z_{673})$. We are given the second and third coefficient of the polynomial, so it is worth trying to use Vieta's Formulas. Firstly, we see that

$$z_1 + z_1 + z_1 + z_2 + \ldots + z_{673} + z_{673} + z_{673} = -20 \Rightarrow z_1 + z_2 + \ldots + z_{673} = -\frac{20}{3}$$

Now to use the second Vieta Formula, recall that we need to find all possible pairs of roots. This means that the sum will include only the square of a single root or the product of two distinct roots. For the former, we can examine z_1^2. There are a total of 3 ways to select 2 z_1's from $(x-z_1)(x-z_1)(x-z_1)$, so the overall sum has 3 instances of z_1^2. For the latter, we can examine z_1z_2. There are a total of $3 \cdot 3 = 9$ ways to select a z_1 and z_2 from $(x-z_1)(x-z_1)(x-z_1)(x-z_2)(x-z_2)(x-z_2)$, so there are 9 instances of z_1z_2. Therefore,

$$3(z_1^2 + z_2^2 + \ldots + z_{673}^2) + 9(z_1z_2 + z_1z_3 + \ldots + z_{672}z_{673}) = 19$$

Note that $z_1z_2 + z_1z_3 + \ldots + z_{672}z_{673} = S$, so we can rewrite this as

$$3(z_1^2 + z_2^2 + \ldots + z_{673}^2) + 9S = 19$$

Now, we can proceed similarly to previous problems. Square the first Vieta Formula to see that

$$\begin{aligned}(z_1 + z_2 + \ldots + z_{673})^2 &= z_1^2 + z_2^2 + \ldots + z_{673}^2 + 2(z_1z_2 + z_1z_3 + \ldots + z_{672}z_{673}) \\ &= z_1^2 + z_2^2 + \ldots + z_{673}^2 + 2S = \frac{400}{9}\end{aligned}$$

To finish, we just substitute this in to get

$$\begin{aligned}3(z_1^2 + z_2^2 + \ldots + z_{673}^2) + 9(z_1z_2 + z_1z_3 + \ldots + z_{672}z_{673}) &= 19 \\ 3\left(\frac{400}{9} - 2S\right) + 9S &= 19 \\ 3S &= -\frac{343}{3} \\ \boxed{S = -\frac{343}{9}}\end{aligned}$$

□

§5 Newton Sums (Optional)

Problem 3.4 was very unpleasant to do. Newton Sums exist to make finding the sum of powers of roots faster and easier (though not by much).

Theorem 5.1 (Newton Sums)
Let the roots of the polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be $x_1, x_2, \ldots, x_n$ and let $P_k = x_1^k + x_2^k + \ldots + x_n^k$ where k is a positive integer. Then the following holds where $a_m = 0$ when $m < 0$:

$$\begin{aligned}0 &= 1a_{n-1} + P_1a_n \\ 0 &= 2a_{n-2} + P_1a_{n-1} + P_2a_n \\ 0 &= 3a_{n-3} + P_1a_{n-2} + P_2a_{n-1} + P_3a_n \\ &\vdots \\ 0 &= ka_{n-k} + P_1a_{n-k+1} + \ldots + P_{k-1}a_{n-1} + P_ka_n.\end{aligned}$$

The statement for Newton Sums looks confusing, but it really comes down to solving several linear equations. This is easier to see through practice.

Problem 5.2 — Let the roots of the polynomial $2x^3 - 6x^2 + 10x + 16$ be a, b, c. Compute $a^3 + b^3 + c^3$.

Solution. While we could use Vieta's Formulas again, let's try Newton Sums. We want to find P_3. The first equation is equivalent to $0 = -6 + 2P_1$ so $P_1 = 3$. Plugging this into the second equation yields $0 = 20 - 18 + 2P_2$ which means $P_2 = -1$. Lastly, we get that $0 = 48 + 30 + 6 + 2P_3$ so we find that $P_3 = \boxed{-42}$ as desired. □